

# MORE ON $SOP_1$ AND $SOP_2$

SAHARON SHELAH AND ALEX USVYATSOV

**ABSTRACT.** This paper continues [DjSh692]. We present a rank function for  $NSOP_1$  theories and give an example of a theory which is  $NSOP_1$  but not simple. We also investigate the connection between maximality in the ordering  $<^*$  among complete first order theories and the  $(N)SOP_2$  property. We complete the proof started in [DjSh692] of the fact that  $<^*$ -maximality implies  $SOP_2$  and get weaker results in the other direction. The paper provides a step toward the classification of unstable theories without the strict order property.

## 1. INTRODUCTION AND PRELIMINARIES

This paper continues [DjSh692] and investigates theories that have or do not have the order properties  $SOP_1$  and  $SOP_2$ . These properties were defined in [DjSh692] in order to find more division lines lying between the tree property (non-simplicity) and  $SOP_3$ , the first dividing line in Shelah's hierarchy of finite approximations of the strict order property. We remind the definitions:

Let  $T$  be a complete first order theory,  $\mathfrak{C}$  - the monster model of  $T$  (a  $\kappa^*$  - saturated and homogeneous model for  $\kappa^*$  big enough).

**Definition 1.1.** (1) Let  $n \geq 3$ . We say  $\varphi(\bar{x}, \bar{y})$  (with  $\text{len}(x) = \text{len}(y)$ ) exemplifies the strong order property of order  $n$  ( $SOP_n$ ) in  $T$  if it defines on  $\mathfrak{C}$  a graph with infinite indiscernible chains with no cycles of length  $n$ .  
 (2) We say  $\varphi(\bar{x}, \bar{y})$  (with  $\text{len}(x) = \text{len}(y)$ ) exemplifies the *strict order property* in  $T$  if it defines on  $\mathfrak{C}$  a partial order with infinite indiscernible chains.

**Fact 1.2.** For a theory  $T$ ,  $\text{strict order property} \implies SOP_{n+1} \implies SOP_n$  for all  $n \geq 3$ .

*Proof.* The first implication is trivial, for the other one see [Sh500], claim (2.6).  $\square$

We also remind an equivalent definition of  $SOP_3$ :

**Fact 1.3.**  $T$  has  $SOP_3$  if and only if there is an indiscernible sequence  $\langle \bar{a}_i : i < \omega \rangle$  and formulae  $\varphi(\bar{x}, \bar{y})$ ,  $\psi(\bar{x}, \bar{y})$  such that

- (a)  $\{\varphi(\bar{x}, \bar{y}), \psi(\bar{x}, \bar{y})\}$  is contradictory,
- (b) for some sequence  $\langle \bar{b}_j : j < \omega \rangle$  we have

$$i \leq j \implies \models \varphi[\bar{b}_j, \bar{a}_i] \text{ and } i > j \implies \models \psi[\bar{b}_j, \bar{a}_i].$$

- (c) for  $i < j$ , the set  $\{\varphi(\bar{x}, \bar{a}_j), \psi(\bar{x}, \bar{a}_i)\}$  is contradictory.

*Proof.* Easy, or see [Sh500], claim (2.20).  $\square$

Now we recall the definitions of  $SOP_1$ ,  $SOP_2$  and related properties:

**Definition 1.4.** (1)  $T$  has  $SOP_2$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this property in  $\mathfrak{C}$ , and this means:

There are  $\bar{a}_\eta \in \mathfrak{C}$  for  $\eta \in {}^\omega 2$  such that

- (a) For every  $\eta \in {}^\omega 2$ , the set  $\{\varphi(\bar{x}, \bar{a}_{\eta \upharpoonright l}) : l < \omega\}$  is consistent.
- (b) If  $\eta, \nu \in {}^{\omega>} 2$  are incomparable,  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$  is inconsistent.
- (2)  $T$  has  $SOP_1$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this in  $\mathfrak{C}$ , which means:
  - There are  $\bar{a}_\eta \in \mathfrak{C}$ , for  $\eta \in {}^{\omega>} 2$  such that:
    - (a) for  $\rho \in {}^\omega 2$  the set  $\{\varphi(\bar{x}, \bar{a}_{\rho \upharpoonright n}) : n < \omega\}$  is consistent.
    - (b) if  $\nu \frown \langle 0 \rangle \leq \eta \in {}^{\omega>} 2$ , then  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_{\nu \frown \langle 1 \rangle})\}$  is inconsistent.
- (3)  $NSOP_2$  and  $NSOP_1$  are the negations of  $SOP_2$  and  $SOP_1$  respectively.
- (4)  $T$  has  $SOP'_1$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this property in  $\mathfrak{C}$ , and this means:
  - there are  $\langle \bar{a}_\eta : \eta \in {}^{\omega>} 2 \rangle$  in  $\mathfrak{C}_T$  such that
    - (a)  $\{\varphi(\bar{x}, \bar{a}_{\eta \upharpoonright n})^{n(n)} : n < \omega\}$  is consistent for every  $\eta \in {}^\omega 2$ , where we use the notation

$$\varphi^l = \begin{cases} \varphi & \text{if } l = 1, \\ \neg \varphi & \text{if } l = 0 \end{cases}$$

for  $l < 2$ .

- (b) If  $\nu \frown \langle 0 \rangle \leq \eta \in {}^{\omega>} 2$ , then  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$  is inconsistent.
- (5)  $T$  has  $SOP''_2$  if there is a formula  $\varphi(\bar{x}, \bar{y})$  which exemplifies this property in  $\mathfrak{C}$ , and this means:
  - there is a sequence
 
$$\langle \bar{a}_{\bar{\eta}} : \bar{\eta} = \langle \eta_0, \dots, \eta_{n-1} \rangle, \eta_0 \triangleleft \eta_1 \triangleleft \dots \triangleleft \eta_{n-1} \in {}^{\lambda>} 2 \text{ and } \lg(\eta_i) \text{ successor} \rangle$$
 such that
    - (a) for each  $\eta \in {}^\lambda 2$ , the set
 
$$\left\{ \begin{array}{l} \varphi(x, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta \upharpoonright (\alpha_0 + 1), \eta \upharpoonright (\alpha_1 + 1), \dots, \eta \upharpoonright (\alpha_{n-1} + 1) \rangle \\ \text{and } \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \lambda \end{array} \right\}$$
 is consistent
    - (b) for every large enough  $m$ , if  $h$  is a 1-to-1 function from  ${}^{n \geq m}$  into  ${}^{\lambda>} 2$  preserving  $\eta \triangleleft \nu$  and  $\nu \perp \nu$  (incomparability) then  $\{\varphi(\bar{x}, \bar{a}_{\bar{\nu}}) : \text{for some } \eta \in {}^n m \text{ we have } \bar{\nu} = \langle h(\eta \upharpoonright \ell) : \ell \leq n \rangle\}$  is inconsistent.

**Fact 1.5.** (1) For a theory  $T$ ,  $SOP_3 \implies SOP_2 \implies SOP_1$   
 (2)  $T$  has  $SOP_1$  if and only if it has  $SOP'_1$

*Proof.* See [DjSh692]. □

It is still not known whether the implications in 1.5, (1) are strict, but for now we investigate each one of these order properties on its own.

In the second section we expand our knowledge on  $SOP_1$ . We present a rank function measuring type-definable “squares”, i.e. pairs of types of the form  $(p(\bar{x}), q(\bar{y}))$  and show the rank is finite for every such a pair if and only if  $T$  does not have  $SOP'_1$  (if and only if  $T$  does not have  $SOP_1$ ). In fact, if one calls a tree of parameters  $\{\bar{a}_\eta : \eta \in {}^{\omega>} 2\}$  showing that  $\varphi(\bar{x}, \bar{y})$  exemplifies  $SOP'_1$  in  $\mathfrak{C}$  (as in the definition of  $SOP'_1$ ) a  $\varphi$ - $SOP'_1$  tree, the rank measures exactly the maximal depth of a tree like this that can be built in  $\mathfrak{C}$ . We also show a small application of the rank.

It is easy to see (see [DjSh692]) that if  $\varphi(\bar{x}, \bar{y})$  exemplifies  $SOP_1$  in  $\mathfrak{C}$  then it also exemplifies the tree property, so  $T$  has  $SOP_1 \implies T$  is not simple. We show that the implication is proper, i.e. find an example of a theory  $T$  which is not simple, but is  $NSOP_1$ . This theory which we call  $T_{\text{feq}}^*$ , was first defined in [Sh457],

and is used in [Sh500] as an example of an  $NSOP_3$  non-simple theory. Here we use a slightly different definition of the same theory, as given in [DjSh692].

**Definition 1.6.** (1)  $T_{\text{feq}}$  is the following theory in the language  $\{Q, P, E, R, F\}$

- (a) Predicates  $P$  and  $Q$  are unary and disjoint, and  $(\forall x) [P(x) \vee Q(x)]$ ,
- (b)  $E$  is an equivalence relation on  $Q$ ,
- (c)  $R$  is a binary relation on  $Q \times P$  such that

$$[x R z \ \& \ y R z \ \& \ x E y] \implies x = y.$$

(so  $R$  picks for each  $z \in Q$  (at most one) representative of any  $E$ -equivalence class).

(d)  $F$  is a (partial) binary function from  $Q \times P$  to  $Q$ , which satisfies

$$F(x, z) \in Q \ \& \ (F(x, z)) R z \ \& \ x E (F(x, z)).$$

(so for  $x \in Q$  and  $z \in P$ , the function  $F$  picks the representative of the  $E$ -equivalence class of  $x$  which is in the relation  $R$  with  $z$ ).

(2)  $T_{\text{feq}}^*$  is the model completion of  $T_{\text{feq}}$ , (so a complete theory with infinite models, in which  $F$  is a full function).

If the reader thinks about the definition above, he'll find out that  $T_{\text{feq}}^*$  is just the model completion of the theory of infinitely many (independent) parametrised equivalence relations. The reader can also compare between the definition of  $T_{\text{feq}}^*$  here and in [Sh457]. As we already mentioned, it was shown in [Sh500] this theory does not have  $SOP_3$  (but is not simple). Here we prove an (a priori) stronger result:  $T_{\text{feq}}^*$  does not have  $SOP_1$ .

In the third section we deal with  $\triangleleft_\lambda^*$ -maximality (see the beginning of the section for definitions). For a theory  $T$ , to be  $\triangleleft_\lambda^*$ -maximal means to be complicated. In a way, it means that it is hard to make its models  $\lambda$ -saturated. In [Sh500] it was stated that  $SOP_3$  implies  $\triangleleft_\lambda^*$ -maximality; here we fill the missing details of the proof, showing explicitly that the model completion of the theory of trees is  $\triangleleft_\lambda^*$ -maximal for every regular  $\lambda$  big enough.

We are interested in this paper, though, in  $SOP_2$  more than  $SOP_3$ . In [DjSh692] it was shown that a property similar to  $\triangleleft_\lambda^*$ -maximality (which also follows from  $\triangleleft_\lambda^*$ -maximality for some  $\lambda$  under certain set theoretic conditions) implies  $SOP_2''$ , and one of the questions asked there is what is the connection between  $SOP_2''$  and the  $SOP_n$  hierarchy. Of course, it would be natural to try to connect between  $SOP_2''$  and  $SOP_2$ , and indeed we prove here these two properties are equivalent for a theory  $T$  (not necessarily for a formula).

So we can conclude  $SOP_3 \implies \triangleleft_\lambda^*$ -maximality  $\implies SOP_2$ . Unfortunately, we don't know much about the other directions of the above implications.

In [DjSh692] two notions of “tree indiscernibility” were defined. We recall the definitions:

**Definition 1.7.** (1) Given an ordinal  $\alpha$  and sequences  $\bar{\eta}_l = \langle \eta_0^l, \eta_1^l, \dots, \eta_{n_l}^l \rangle$  for  $l = 0, 1$  of members of  ${}^\alpha 2$ , we say that  $\bar{\eta}_0 \approx_1 \bar{\eta}_1$  iff

- (a)  $n_0 = n_1$ ,
- (b) the truth values of

$$\eta_{k_3}^l \leq \eta_{k_1}^l \cap \eta_{k_2}^l, \quad \eta_{k_1}^l \cap \eta_{k_2}^l \triangleleft \eta_{k_3}^l, \quad (\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 0 \rangle \leq \eta_{k_3}^l,$$

for  $k_1, k_2, k_3 \leq n_0$ , do not depend on  $l$ .

- (2) We say that the sequence  $\langle \bar{a}_\eta : \eta \in {}^{\alpha>}2 \rangle$  of  $\mathfrak{C}$  (for an ordinal  $\alpha$ ) are *1-fully binary tree indiscernible (1-fbti)* iff whenever  $\bar{\eta}_0 \approx_1 \bar{\eta}_1$  are sequences of elements of  ${}^{\alpha>}2$ , then

$$\bar{a}_{\bar{\eta}_0} =: \bar{a}_{\eta_0^0} \frown \dots \frown \bar{a}_{\eta_{n_0}^0}$$

and the similarly defined  $\bar{a}_{\bar{\eta}_1}$ , realize the same type in  $\mathfrak{C}$ .

- (3) We replace 1 by 2 in the above definition iff  $(\eta_{k_1}^l \cap \eta_{k_2}^l) \frown \langle 0 \rangle \leq \eta_{k_3}^l$  is omitted from clause (b) above.

We will need the following fact proved in [DjSh692], (2.11):

**Fact 1.8.** *If  $t \in \{1, 2\}$  and  $\langle \bar{b}_\eta : \eta \in {}^{\omega>}2 \rangle$  are given, and  $\delta \geq \omega$ , then we can find  $\langle \bar{a}_\eta : \eta \in {}^{\delta>}2 \rangle$  such that*

- (a)  $\langle \bar{a}_\eta : \eta \in {}^{\delta>}2 \rangle$  is *t-fbti*,
- (b) if  $\bar{\eta} = \langle \eta_m : m < n \rangle$ , where each  $\eta_m \in {}^{\delta>}2$  is given, and  $\Delta$  is a finite set of formulae of  $T$ , then we can find  $\nu_m \in {}^{\omega>}2$  ( $m < n$ ) such that with  $\bar{\nu} =: \langle \nu_m : m < n \rangle$ , we have  $\bar{\nu} \approx_t \bar{\eta}$  and the sequences  $\bar{a}_{\bar{\eta}}$  and  $\bar{b}_{\bar{\nu}}$ , realise the same  $\Delta$ -types.

## 2. MORE ON $SOP_1$

We work with a complete first order theory  $T$ , let  $\mathfrak{C}$  be its “monster” model (saturated in some very big  $\kappa^*$ ). Let  $\mathcal{L} = \mathcal{L}(T)$  (the language of  $T$ ). Every formula we mention in this section is an  $\mathcal{L}$ -formula, maybe with parameters from  $\mathfrak{C}$ .

First, we would like to make sure that we indeed are developing a new theory here. As every simple theory is  $NSOP_1$ , it is very important to ask whether the other direction is also true (if so, we would find ourselves in a well-developed context, for which almost all the theorems proven here are either known or easy). But the answer is negative:

**Theorem 2.1.**  *$T_{feq}^*$  does not have  $SOP_1$ .*

*Proof.* Suppose there exists  $\varphi(\bar{x}, \bar{y})$  with  $\ell g(\bar{x}) = n$ ,  $\ell g(\bar{y}) = m$ , and  $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$  in  ${}^m\mathfrak{C}$  which exemplify  $SOP_1$  in  $\mathfrak{C}$  ( $\mathfrak{C}$  is the monster model of  $T_{feq}^*$ ). Without loss of generality, (by ref{fct:thinning})  $\langle \bar{a}_\eta : \eta \in {}^{\omega>}2 \rangle$  if 1-full tree indiscernible. Also, by elimination of quantifiers, we may assume that  $\varphi(\bar{x}, \bar{y})$  is quantifier free. As the only function symbol in the language is  $F$  and  $F^\mathfrak{C}$  has the property  $F^\mathfrak{C}(F^\mathfrak{C}(x, z), y) = F^\mathfrak{C}(x, y)$  for all  $z$ , we will also assume wlog that  $\bar{x}$  and  $\bar{y}$  in  $\varphi(\bar{x}, \bar{y})$  are closed under  $F$  and  $\varphi(\bar{x}, \bar{y})$  gives the full diagram of  $\bar{x} \frown \bar{y}$ . We shall regard  $\bar{x}$  as  $\langle x^0, \dots, x^{n-1} \rangle$ ,  $\bar{y}$  as  $\langle y^0, \dots, y^{m-1} \rangle$ ,  $\bar{a}_\eta$  as  $\langle a_\eta^0, \dots, a_\eta^{m-1} \rangle$ .

By the definition of  $SOP_1$ , there exist  $\bar{e} = \langle e^0, \dots, e^{n-1} \rangle$ ,  $\bar{d} = \langle d^0, \dots, d^{n-1} \rangle$  in  ${}^n\mathfrak{C}$  s.t.

$$\mathfrak{C} \models \varphi(\bar{e}, \bar{a}_{\langle \rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 0 \rangle}) \wedge \varphi(\bar{e}, \bar{a}_{\langle 00 \rangle})$$

and

$$\mathfrak{C} \models \varphi(\bar{d}, \bar{a}_{\langle \rangle}) \wedge \varphi(\bar{d}, \bar{a}_{\langle 1 \rangle})$$

Denote  $\eta = \langle 00 \rangle$ . Let  $B = \mathfrak{C} \upharpoonright \bar{a}_\eta \frown \bar{a}_{\langle 1 \rangle}$ . By our assumptions, there exists a model  $N_0$  whose universe is  $\bar{x} \frown \bar{a}_\eta$ , extending  $\mathfrak{C} \upharpoonright \bar{a}_\eta$ , whose basic diagram is  $\varphi(\bar{x}, \bar{a}_\eta)$ . Similarly, there exists a model  $N_1$  with universe  $\bar{x} \frown \bar{a}_{\langle 1 \rangle}$  and basic diagram  $\varphi(\bar{x}, \bar{a}_{\langle 1 \rangle})$ . We shall amalgamate  $B, N_0$  and  $N_1$  into a model of  $T_{feq}$ ,  $N$ . This will immediately give a contradiction: first, extend  $N$  to  $N^* \models T_{feq}^*$ , then

amalgamate  $N^*$  and  $\mathfrak{C}$  over  $B$  into some  $\mathfrak{C}^+ \models T_{feq}^*$ . By model completeness of  $T_{feq}^*$ ,  $\mathfrak{C} \prec \mathfrak{C}^+$ , but  $\mathfrak{C}^+ \models \exists \bar{x}(\varphi(\bar{x}, \bar{a}_\eta) \wedge \varphi(\bar{x}, \bar{a}_{\langle 1 \rangle}))$ , which is a contradiction to the definition of  $SOP_1$ .

It is left, therefore, to show that we can define on  $|N_0| \cup |N_1|$  a structure which will be a model of  $T_{feq}$ , extending  $B$ .

We define  $N$  as follows:

$$|N| = |N_1| \cup |N_2|, \quad P^N = P^{N_1} \cup P^{N_2}, \quad Q^N = Q^{N_1} \cup Q^{N_2}.$$

Note that the diagram of  $\bar{x}$  in  $N_0$  is the same as the diagram of  $\bar{x}$  in  $N_1$  (both implied by  $\varphi(\bar{x}, \bar{y})$ , and the diagrams of  $\bar{a}_\eta, \bar{a}_{\langle 1 \rangle}$  in  $N_i$  are the same as in  $\mathfrak{C}$ , hence the same as in  $B$ . Therefore,  $P^N$  and  $Q^N$  are well defined and give a partition of  $|N|$ . Also, so far  $N$  extends  $B$  (as a structure).

Considering  $E$  and  $R$ , we define

$$\begin{aligned} R^N &= R^{N_1} \cup R^{N_2} \cup R^B \\ E^N &= E^{N_1} \cup E^{N_2} \cup E^B \end{aligned}$$

Once we have proven the following lemmas, we will be able to define  $F^N$  in a natural way, and in fact will be done.

*Lemma 2.1.1.*  $E^N$  is an equivalence relation on  $Q^N$ , extending  $E^B$ .

*Lemma 2.1.2.*  $R^N$  is a two-place relation on  $N$ ,  $R^N \subseteq P^N \times Q^N$ , satisfying:

for every  $y \in P^N$  and every equivalence class  $C$  of  $E^N$ , there exists a unique  $z \in C$  such that  $(y, z) \in R^N$ .

*Proof of 2.1.1.* The only nonobvious thing is transitivity. We check two main cases, all the rest are either similar or trivial.

- (1) Assume  $x^i E^N a_\eta^j$ ,  $x^i E^N a_{\langle 1 \rangle}^k$  for some  $i, j, k$ . We want to show  $a_\eta^j E^N a_{\langle 1 \rangle}^k$ .

It is enough to see  $a_\eta^j E^{\mathfrak{C}} a_{\langle 1 \rangle}^k$ . We will write  $E$  instead of  $E^{\mathfrak{C}}$ .

$N \models x^i E a_\eta^j \Rightarrow N_0 \models x^i E a_\eta^j \Rightarrow \varphi(\bar{x}, \bar{y}) \vdash x^i E y^j$ . Similarly,  $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^k$ , and we get (by the choice of  $\bar{e}, \bar{d} \in {}^n \mathfrak{C}$ )

$$e^i E a_\eta^j, e^i E a_{\langle 1 \rangle}^k, e^i E a_\eta^j, e^i E a_{\langle 1 \rangle}^k, d^i E a_\eta^j, d^i E a_{\langle 1 \rangle}^k, d^i E a_{\langle 1 \rangle}^j, d^i E a_{\langle 1 \rangle}^k.$$

Now it is easy to see that all the above elements are  $E$ -equivalent in  $\mathfrak{C}$ , in particular  $a_\eta^j$  and  $a_{\langle 1 \rangle}^k$ , as required.

- (2) Assume  $x^i E^N a_\eta^n$ ,  $a_{\langle 1 \rangle}^k E^N a_\eta^n$ , and we show  $x^i E^N a_{\langle 1 \rangle}^k$ , i.e.  $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^k$ . As  $\varphi(\bar{d}, \bar{a}_{\langle 1 \rangle})$  holds in  $\mathfrak{C}$  and  $\varphi(\bar{x}, \bar{y})$  gives a full diagram, it will be enough to see  $d^i E a_{\langle 1 \rangle}^k$ .

We know that  $\varphi(\bar{x}, \bar{y}) \vdash x^i E y^j$  therefore  $e^i E a_\eta^j$ ,  $e^i E a_{\langle 1 \rangle}^j$ ,  $d^i E a_{\langle 1 \rangle}^j$ ,  $d^i E a_{\langle 1 \rangle}^j$ . In particular,  $d^i E a_\eta^j$ , but, by our assumption,  $a_\eta^j E a_{\langle 1 \rangle}^k$ , so we are done.

□<sub>1</sub>

*Proof of 2.1.2.* Like in the previous lemma, the only nontrivial thing to prove is the last part, and we will deal with two main cases.

- (1)  $N \models (a_\eta^i R a_{\langle 1 \rangle}^j) \wedge (a_\eta^i R x^k) \wedge (x^k E a_{\langle 1 \rangle}^j)$ . We aim to show  $N \models (x^k = a_{\langle 1 \rangle}^j)$ .

We know:

$$(*)_1 \quad \mathfrak{C} \models a_\eta^i R a_{\langle 1 \rangle}^j$$

$$(*)_2 \quad N_0 \models a_\eta^i R x^k, \text{ therefore } \varphi(\bar{x}, \bar{y}) \vdash y^i R x^k$$

(\*)<sub>3</sub>  $N_1 \models x^k E a_{\langle 1 \rangle}^j$ , therefore  $\varphi(\bar{x}, \bar{y}) \vdash x^k E y^j$ .

So we can conclude:

(\*)<sub>2</sub>  $\Rightarrow a_{\langle \rangle}^i R e^k, \quad a_{\langle \rangle}^i R d^k$

(\*)<sub>3</sub>  $\Rightarrow e^k E a_{\langle \rangle}^j, \quad d^k E a_{\langle \rangle}^j \Rightarrow e^k E d^k$ .

As the above two relations hold in  $\mathfrak{C}$ , which is a model of  $T_{feq}$ , we get  $\mathfrak{C} \models e^k = d^k$ . Denote  $e^* = e^k = d^k$ .

(\*)<sub>1</sub>  $\Rightarrow a_\eta^i R a_{\langle 1 \rangle}^j$

(\*)<sub>2</sub>  $\Rightarrow a_\eta^i R e^*$

(\*)<sub>1</sub>  $\Rightarrow e^* E a_{\langle 1 \rangle}^j$

Together (once again,  $\mathfrak{C} \models T_{feq}$ ) we get  $e^* = a_{\langle 1 \rangle}^j$ , therefore  $\varphi(\bar{x}, \bar{y}) \vdash x^k = y^j$ , so  $N_1 \models x^k = a_{\langle 1 \rangle}^j$ , and we are done.

(2)  $N \models (x^i R a_{\langle 1 \rangle}^j) \wedge (x^i R a_\eta^k) \wedge (a_\eta^k E a_{\langle 1 \rangle}^j)$  and we aim to show  $N \models (a_\eta^k = a_{\langle 1 \rangle}^j)$ .

We know:

(\*)<sub>1</sub>  $N_1 \models x^i R a_{\langle 1 \rangle}^j$ , so  $\varphi(\bar{x}, \bar{y}) \vdash x^i R y^j$

(\*)<sub>2</sub>  $N_0 \models x^i R a_\eta^k$ , so  $\varphi(\bar{x}, \bar{y}) \vdash x^i R y^k$

(\*)<sub>3</sub>  $\mathfrak{C} \models a_\eta^k E a_{\langle 1 \rangle}^j$

Note that by indiscernibility of  $\langle \bar{a}_r : r \in {}^{w>2} \rangle$  and (\*)<sub>3</sub> we get  $a_{\langle 0 \rangle}^k E a_{\langle 1 \rangle}^j$ , therefore  $a_{\langle 0 \rangle}^k E a_\eta^k$ . Now, by (\*)<sub>2</sub>,  $e^i R a_\eta^k$  &  $e^i R a_{\langle 0 \rangle}^k$ . Therefore, by  $\mathfrak{C} \models T_{feq}$ ,  $a_{\langle 0 \rangle}^k = a_\eta^k$ . Now by indiscernibility

$$a_{\langle 0 \rangle}^k = a_{\langle \rangle}^k, \quad a_{\langle 1 \rangle}^k = a_{\langle \rangle}^k$$

So we get that all of the above are equal (and in fact  $a_{r_1}^k = a_{r_2}^k$  for all  $r_1, r_2 \in {}^{w>2}$ ).

Now:

(\*)<sub>1</sub>  $\Rightarrow d^i R a_{\langle 1 \rangle}^j$

(\*)<sub>2</sub>  $\Rightarrow d^i R a_{\langle 1 \rangle}^k \Rightarrow d^i R a_\eta^k$  (as  $a_{\langle 1 \rangle}^k = a_\eta^k$ )

(\*)<sub>3</sub>  $\Rightarrow a_\eta^k E a_{\langle 1 \rangle}^j$ .

By  $\mathfrak{C} \models T_{feq}$ , we conclude  $a_\eta^k = a_{\langle 1 \rangle}^j$ , which finishes the proof of the lemma, and therefore the proof of the theorem.  $\square_2$

$\square$

Our next goal is to show that there is a rank function closely connected with being (N)SOP<sub>1</sub>. Let  $\varphi \bar{x}, \bar{y}$  be a formula.

**Definition 2.2.** Given (partial) types  $p(\bar{x}), q(\bar{y})$ . By induction on  $n < \omega$  we define when

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) \geq n :$$

If  $\underline{n} = 0$ , this happens if both  $p(\bar{x}), q(\bar{y})$  are consistent

For  $\underline{n+1}$ , the rank is  $\geq n+1$  if for some  $\bar{c} \models q(\bar{y})$ , both

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}, q(\bar{y})) \geq n$$

and

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}) \geq n.$$

We say  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) = \infty$  iff  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) \geq n$  for all  $n$ .

We say the rank is  $-1$  if it is not bigger or equal to 0.

*Remark 2.3.* (1) The statement  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(\theta_1(\bar{x}; \bar{a}), \theta_2(\bar{x}; \bar{b})) \geq n$  is a first order formula with parameters  $\bar{a}, \bar{b}$ .

(2) We can continue to define when  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) \geq \alpha$  for any ordinal  $\alpha$ , but by the compactness theorem and part (1) it follows that  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p, q) \geq \alpha$  for some  $\alpha \geq \omega$  iff  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p, q) \geq \omega$  iff  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p, q) = \infty$ .

(3) (Monotonicity) If  $p' \vdash p''$  and  $q' \vdash q''$ , then  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p', q') \leq \text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p'', q'')$ .

(4) (Finite Character) If  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p(\bar{x}), q(\bar{y})) = n$ , then for some finite  $p_0(\bar{x}) \subseteq p(\bar{x})$  and  $q_0(\bar{y}) \subseteq q(\bar{y})$  we have  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0(\bar{x}), q_0(\bar{y})) = n$ .

(5) If  $p' \equiv p''$ , and  $q' \equiv q''$ , then  $\text{rk}_{\varphi}^1(p', q') = \text{rk}_{\varphi}^1(p'', q'')$ .

We aim to show that  $\text{rk}_{\varphi}^1(p(\bar{x}), q(\bar{y}))$  is finite for every  $p(\bar{x}), q(\bar{y})$  (or, equivalently,  $\text{rk}_{\varphi}^1(\bar{x} = \bar{x}, \bar{y} = \bar{y})$  is finite) if and only if  $\varphi(\bar{x}, \bar{y})$  does not exemplify  $SOP_1'$  in  $T$ . For this purpose we shall need another definition and several easy claims.

**Definition 2.4.** Given (partial) types  $p(\bar{x})$  and  $q(\bar{y})$ , we say that  $\{\bar{a}_\eta : \eta \in {}^{n \geq 2}\}$  is a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x})$  and  $q(\bar{y})$  (of depth  $n$ ) if

- (a)  $p(\bar{x}) \cup \{\varphi^{\eta(i)}(\bar{x}, \bar{a}_{\eta \upharpoonright i}) : i < n\}$  is consistent for every  $\eta \in {}^{n \geq 2}$ .
- (b)  $\bar{a}_\eta \models q(\bar{y})$  for all  $\eta \in {}^{n \geq 2}$
- (c) If  $\eta, \nu$  are in  ${}^{n \geq 2}$  satisfying  $\eta \frown \langle 0 \rangle \leq \nu$ , then the set  $\{\varphi(\bar{x}, \bar{a}_\eta), \varphi(\bar{x}, \bar{a}_\nu)\}$  is inconsistent.

**Proposition 2.5.** Suppose  $\{\bar{a}_\eta : \eta \in {}^{n \geq 2}\}$  is a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x})$  and  $q(\bar{y})$  of depth  $n$ , and denote  $A^0 = \{\bar{a}_\eta : \langle 0 \rangle \leq \eta\}$ ,  $A^1 = \{\bar{a}_\eta : \langle 1 \rangle \leq \eta\}$ . Then

- (1)  $A^1$  is a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{a}_{\langle 1 \rangle})\}$  and  $q(\bar{y})$
- (2)  $A^0$  is a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x})$  and  $q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{a}_{\langle 1 \rangle}))\}$ .

*Proof.* The clauses (a) and (c) of the definition easily hold both for  $A^1$  and  $A^0$ , so we should only check (b), which is also obvious for  $A^1$ . Therefore, we're left to show that for every  $\eta \in A^0$ ,  $\bar{a}_\eta \models \neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{a}_{\langle 1 \rangle}))$ , and this is clear by clause (c) of the definition ( $\{\bar{a}_\eta : \eta \in {}^{n \geq 2}\}$  is a  $\varphi$ - $SOP_1'$  tree, and  $\langle 1 \rangle \frown 0 \leq \eta$ ).  $\square$

Now we show the connection between the rank and  $SOP_1'$  trees.

**Proposition 2.6.**  $\text{rk}_{\varphi}^1(p(\bar{x}), q(\bar{y})) \geq n \iff$  there exists a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x})$  and  $q(\bar{y})$  of depth  $n$ .

*Proof.* Both directions are proved by induction on  $n$ . The case  $n = 0$  is obvious. For  $n = m + 1$ , the right-to-left direction follows immediately by the induction hypothesis and 2.5. So we will elaborate more only about the other direction, although it is also straightforward.

Suppose  $n = m + 1$  and  $\text{rk}_{\varphi}^1(p(\bar{x}), q(\bar{y})) \geq n$ . By the definition of the rank and the induction hypothesis, for some  $\bar{c} \models q(\bar{y})$ , there are

- (1) a  $\varphi$ - $SOP_1'$  tree  $A^1 = \{\bar{a}_\eta^1 : \eta \in {}^{m \geq 2}\}$  for  $p(\bar{x}) \cup \{\varphi(\bar{x}, \bar{c})\}$  and  $q(\bar{y})$
- (2) a  $\varphi$ - $SOP_1'$  tree  $A^0 = \{\bar{a}_\eta^0 : \eta \in {}^{m \geq 2}\}$  for  $p(\bar{x})$  and  $q(\bar{y}) \cup \{\neg(\exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c}))\}$

(both of depth  $m$ ). We define a tree  $\{\bar{a}_\eta : \eta \in {}^{n \geq 2}\}$  by

$$\begin{aligned} \bar{a}_{\langle 1 \rangle} &= \bar{c} \\ \bar{a}_{\langle \ell \rangle \frown \eta} &= \bar{a}_\eta^\ell \text{ for } \ell \in \{0, 1\} \end{aligned}$$

which is as required, i.e. a  $\varphi$ - $SOP_1'$  tree for  $p(\bar{x})$  and  $q(\bar{y})$ . Why?

- (a) of the definition obviously holds by (1) above.

- (b) holds as  $\bar{c} \models q(\bar{y})$ .
- (c) obviously holds by (2) above.

□

The following remark is obvious:

*Remark 2.7.*  $\varphi(\bar{x}, \bar{y})$  exemplifies  $SOP'_1$  in  $T \iff$  there exists a  $\varphi$ - $SOP'_1$  tree for  $\bar{x} = \bar{x}$  and  $\bar{y} = \bar{y}$  of any depth.

So we can conclude the following

**Theorem 2.8.** *A formula  $\varphi(\bar{x}, \bar{y})$  does not exemplify  $SOP'_1$  in  $T \iff \text{rk}_\varphi^1(\bar{x} = \bar{x}, \bar{y} = \bar{y}) < \omega \iff \text{rk}_\varphi^1(p(\bar{x}), q(\bar{y})) < \omega$  for every two (partial) types  $p(\bar{x})$  and  $q(\bar{y})$ . Moreover,  $\text{rk}_\varphi^1(\bar{x} = \bar{x}, \bar{y} = \bar{y})$  is exactly the maximal depth of a  $\varphi$ - $SOP'_1$  tree that can be built in  $\mathfrak{C}$ .*

**Corollary 2.9.**  *$T$  does not have  $SOP_1 \iff T$  does not have  $SOP'_1 \iff \text{rk}_\varphi^1(\bar{x} = \bar{x}, \bar{y} = \bar{y})$  is finite for every formula  $\varphi(\bar{x}, \bar{y})$ .*

Now we show an application of the rank.

**Theorem 2.10.** *Suppose that  $T$  satisfies  $NSOP_1$ . Assume that*

- (a)  $M_1 \prec M_2 \prec \mathfrak{C}$ .
- (b)  $p$  is a (not necessarily complete) type over  $M_2$ , containing the formula  $\varphi(\bar{x}, \bar{b}^*)$  for some  $\bar{b}^* \in M^2 \setminus M^1$ .

*Then for some finite  $q' \subseteq \text{tp}(\bar{b}^*/M_1)$  at least one of the following holds:*

- (i) *If  $\bar{b} \in M_1$  realises  $q'(\bar{y})$  then  $\varphi(\bar{x}, \bar{b}) \notin p$ , or*
- (ii) *If  $\bar{b} \in M_1$  realises  $q'(\bar{y})$  then  $\{\varphi(\bar{x}, \bar{b}), \varphi(\bar{x}, \bar{b}^*)\}$  is consistent.*

*In fact, all we need to assume for this Claim is that  $\varphi(\bar{x}, \bar{y})$  does not exemplify that  $T$  is  $SOP_1$ .*

*Proof.* Denote  $q = \text{tp}(\bar{b}^*/M_1)$ . As  $T$  is  $NSOP_1$ , we have that  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p \upharpoonright M_1, q) = n^* < \omega$  (certainly  $n^* \geq 0$ ). By the finite character of the rank, we have that for some finite  $p_0 \subseteq p \upharpoonright M_1$  and  $q_0 \subseteq q$ ,

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0, q_0) = n^*.$$

Hence for no  $\bar{c} \models q_0(\bar{y})$  do we have that both  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{c})\}, q_0) \geq n^*$  and  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0, q_0 \cup \{(\neg \exists \bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{c})]\}) \geq n^*$ . In particular, this holds for  $\bar{c} = \bar{b}^*$  (remember that  $\bar{b}^* \models q$  and therefore certainly  $\bar{b}^* \models q_0$ ). So

⊗ 2.10.1.

If  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0) \geq n^*$ , then  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0, q_0 \cup \{(\neg \exists \bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{b}^*)]\}) < n^*$ .

By Remark 2.3(1), there is a finite  $q' \subseteq q$  such that

⊗ 2.10.2.

$$\bar{b} \text{ realises } q' \implies \text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b})\}, q_0) = \text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0).$$

We aim to show that  $q'$  is as required.

**Case 1.**  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0) = n < n^*$ .

We note that the possibility (i) holds.

Namely, suppose  $\bar{b}$  realises  $q'$ , then  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b})\}, q_0) = n < n^*$ , so if  $\varphi(\bar{x}, \bar{b}) \in p$ , we obtain a contradiction with monotonicity of the rank.



Case 2.  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b}^*)\}, q_0) = n^*$ .

We shall show that (ii) holds.

Suppose otherwise, so let  $\bar{b} \in M_1$  realise  $q'$  and  $\{\varphi(\bar{x}, \bar{b}), \varphi(\bar{x}, \bar{b}^*)\}$  is contradictory. By 2.10.2,

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0 \cup \{\varphi(\bar{x}, \bar{b})\}, q_0) = n^*$$

and by 2.10.1,

$$\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p_0, q_0 \cup \{(\neg \exists \bar{x})(\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{b}))\}) < n^*.$$

We have that  $(\neg \exists \bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{b})] \in q$ , hence  $q_0 \cup \{(\neg \exists \bar{x})[\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{b})]\} \subseteq q$ , in contradiction with monotonicity and  $\text{rk}_{\varphi(\bar{x}, \bar{y})}^1(p \upharpoonright M_1, q) = n^*$ .  $\square$

### 3. MORE ON $SOP_2$ , $SOP_3$ AND $\triangleleft_\lambda^*$ -ORDER

We try to find a connection between the syntactic properties  $SOP_2, SOP_3$  and the semantic property of being  $\triangleleft_\lambda^*$ -maximal. Our guess is that  $\triangleleft_\lambda^*$ -maximality should be equivalent to one of the above order properties (maybe both), but all we manage to prove here is  $SOP_3 \Rightarrow \triangleleft_\lambda^*$ -maximality  $\Rightarrow SOP_2$ . We also give a weak “local” result in the other direction.

First we generalize the definitions from [DjSh692], of  $\triangleleft_\lambda^*$ -maximality, making them local as well as global.

**Definition 3.1.** (1) For given (complete first order theories)  $T_1, T_2$  and cardinals  $\lambda \geq \mu > \kappa, \mu \geq \theta > |T_1| + |T_2| + \aleph_0$

(a)  $T_1 \triangleleft_{\lambda, < \mu, \kappa, < \theta}^* T_2$  means that there is a (complete first order theory)  $T^*$  and interpretations  $\bar{\varphi}_1, \bar{\varphi}_2$  of  $T_1, T_2$  in  $T^*$  respectively,  $|T^*| < \theta$  such that:

–  $\boxtimes_{T^*, \bar{\varphi}_1, \bar{\varphi}_2}^{< \lambda, < \mu, \kappa}$  if  $M$  is a  $\kappa$ -saturated model of  $T^*$  and  $M_\ell = M^{[\bar{\varphi}_\ell]}$  for  $\ell = 1, 2$  and  $M_2$  is  $\lambda$ -saturated (model of  $T_2$ ), then  $M_1$  is  $\mu$ -saturated

(b)  $(T_1, \vartheta_1(\bar{x}, \bar{y})) \triangleleft_{\lambda, < \mu, < \kappa}^* (T_2, \vartheta_2(\bar{x}, \bar{y}))$  means that  $\vartheta_\ell(\bar{x}, \bar{y}) \in L(\tau_{T_\ell})$  and that there is a  $T^*$  and interpretations  $\bar{\varphi}_1, \bar{\varphi}_2$  of  $T_1, T_2$  in  $T^*$  respectively,  $|T^*| < \mu$  such that  $\boxtimes_{T^*, \vartheta_1, \vartheta_2, \bar{\varphi}_1, \bar{\varphi}_2}^{< \lambda, < \mu, \kappa}$  if  $M$  is a  $\kappa$ -saturated model of  $T^*$  and  $M_\ell = M^{[\bar{\varphi}_\ell]}$  for  $\ell = 1, 2$  and  $M_2$  is  $(\lambda, \vartheta_1(\bar{x}, \bar{y}))$ -saturated (see 3 below), then  $M_i$  is  $(\mu, \vartheta_2)$ -saturated.

(2) Instead “ $< \lambda^+$ ” we may write “ $\lambda$ ”, and instead “ $< \mu^+$ ” we may write  $\mu$ , instead “ $< \theta^+$ ” we may write  $\theta$ . If we omit  $\mu$  we mean  $\mu = \lambda$ , and if we write  $\kappa = 0$  then “ $\kappa$ -saturated” becomes the empty demand, if we omit  $\theta$  we mean  $|T_1| + |T_2| + \aleph_0$  and if we omit  $\kappa$  and  $\theta$  then we mean that  $\mu = \lambda, \theta = |T_1| + |T_2| + \aleph_0$ .

(3) We say  $M$  is  $(\lambda, \Delta)$ -saturated when: if  $p \subseteq \{\vartheta(\bar{x}; \bar{a}) : \vartheta(\bar{x}; \bar{y}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{y})} M\}$  is finitely satisfiable of cardinality  $< \lambda$  then  $p$  is realized in  $M$ . If  $\Delta = \{\vartheta(\bar{x}, \bar{y})\}$  we may write  $\vartheta(\bar{x}, \bar{y})$  instead of  $\Delta$ .

(4) If  $T_1, T_2$  are not necessarily complete, then above  $T^*$  is not necessarily complete and we demand: if  $M_1 \models T_1, M_2 \models T_2$  then there is  $M \models T^*$  such that  $M^{[\bar{\varphi}_\ell]} \models Th(M_\ell)$  for  $\ell = 1, 2$ .

(5) We say  $T$  is  $\triangleleft_{\lambda, \kappa}^*$ -maximal if  $|T'| < \lambda \Rightarrow T' \triangleleft_{\lambda, \kappa}^* T$ . We say  $(T, \vartheta(\bar{x}; \bar{y}))$  is  $\triangleleft_{\lambda, \kappa}^*$ -maximal if  $|T'| < \lambda \& \vartheta'(\bar{x}'; \bar{y}') \in L(\tau_{T'}) \Rightarrow (T', \vartheta'(\bar{x}'; \bar{y}')) \triangleleft_{\lambda, \kappa}^* (T, \vartheta(\bar{x}; \bar{y}))$ .

- Definition 3.2.** (1)  $T_{tr}$  is the theory of trees (i.e. the vocabulary is  $\{<\}$  and the axioms state that  $<$  is a partial order and  $\{y : y < x\}$  is a linear order for every  $x$ ), so  $T_{tr}$  is not complete, and let  $\vartheta_{tr}(x, y) = (y < x)$ .  
 (2)  $T_{tr}^*$  is the model completion of  $T_{tr}$ .  
 (3)  $T_{ord}$  is the theory of linear orders,  $T_{ord}^*$  is its model completion (i.e. the theory of dense linear order without endpoints).

We note connection to previous works and obvious properties

- Proposition 3.3.** (1)  $T_1 \triangleleft_{\lambda, \mu, 0}^* T_2$  is  $T_1 \triangleleft_{\lambda, \mu}^* T_2$  of [DjSh692].  
 (2)  $T_1 \triangleleft_{\lambda, \lambda; < \kappa}^* T_2$  implies  $T_1 \triangleleft_{\lambda, \kappa}^* T_2$  of [Sh500, 2.x.p.xxx].  
 (3)  $\triangleleft_{\lambda, \mu; \kappa, \theta}^*$  has the obvious monotonicity properties: if  $T_1 \triangleleft_{< \lambda_1, < \mu'_1; < \kappa_1, < \theta_1}^* T_2$  and  $\lambda_2 \geq \lambda_1, \mu_2 \leq \mu_1, \kappa_2 \geq \kappa_1, \theta_2 \geq \theta_1$  then  $T_1 \triangleleft_{< \lambda_2, < \mu_2; < \kappa_2, < \theta_2}^* T_2$ .  
 (4)  $T \triangleleft_{\lambda, \mu; \kappa, \theta}^* T$  if  $|T| < \theta, \lambda \geq \mu > \kappa, \mu \geq \theta$ .  
 (5) If  $\mu$  is a limit cardinal, then  $T_1 \triangleleft_{< \lambda, < \mu; < \kappa, < \theta}^* T_2$  iff for every  $\mu_1 < \mu, \mu_1 \geq \kappa$  we have

$$T_1 \triangleleft_{< \lambda, < \mu_1; < \kappa, < \theta}^* T_2.$$

- (6) Similar results hold for  $(T_\ell, \vartheta_\ell(\bar{x}; \bar{y}))$ .

*Proof.* Easy. □

- Proposition 3.4.** (1) Assume  $T_1 \triangleleft_{< \lambda, < \mu; < \kappa, < \theta}^* T_2$ . Then for any theory  $T^*$ , we can find  $T^{**} \supseteq T^*$  complete  $|T^{**}| < (|T^*|^{\tau(T_1)} + |T^*|^{\tau(T_2)} + \theta)$  such that: for any interpretations  $\bar{\varphi}_1, \bar{\varphi}_2$  of  $T_1, T_2$  in  $T^{**}$  respectively the definition of  $T_1 \triangleleft_{< \lambda, < \mu; < \kappa, < \theta}^* T_2$  holds.  
 (2) Assume  $\tau(T_1), \tau(T_2)$  are disjoint. Then  $T_1 \triangleleft_{< \lambda, < \mu; < \kappa, < \theta}^* T_2$  if for any  $T \supseteq T_1 \cup T_2$  there is  $T^* \supseteq T$  as demanded in Definition for the trivial interpretations  $M^{\{\bar{\varphi}_1\}}$  is the  $\tau(T_\ell)$ -reduct.

*Proof.* Easy. □

Now we will show that  $T_{tr}^*$  is  $\triangleleft_\lambda^*$ -maximal for every  $\lambda$  big enough, and conclude that  $SOP_3 \Rightarrow \triangleleft_\lambda^*$ -maximality. The last result appears already in [Sh500], theorem (2.9), but the proof is not full - in fact, the proof shows the following theorem:

**Theorem 3.5.** Any theory  $T$ ,  $|T| < \lambda$ , with  $SOP_3$  is  $\triangleleft_\lambda^*$ -above  $T_{ord}^*$ .

*Proof.* See [Sh500], (2.12). □

Here we fill the missing part, proving explicitly that  $T_{tr}^*$ , and therefore  $T_{ord}^*$  are maximal.

**Theorem 3.6.**  $T_{tr}^*$  is  $\triangleleft_\lambda^*$ -maximal for any  $\lambda > \aleph_0$ ; the witness  $T^*$  does not depend on  $\lambda$ .

*Remark 3.7.* This continues [Sh:c, Ch.VI,3.x].

*Proof.* Let  $T$  be any complete theory,  $|T| < \lambda$  and  $M_1$  a model of  $T$ .

Let  $\Phi = \{\varphi(x, \bar{a}) : \varphi(x, \bar{y}) \in L(\tau_T), \bar{a} \in {}^{\ell g(\bar{y})}(M_1)\}$ , so  $|\Phi| = \|M_1\|$ . So  $M = (\omega > \Phi, \triangleleft)$  is a model of  $T_{tr}$  and there is a model  $M_2$  of  $T_{tr}^*$  of cardinality  $\|M_1\|$  extending  $M$  such that every member of  $M_2$  is below some member of  $M$ .

Let  $\chi$  be large enough such that  $M_1, M_2 \in \mathcal{H}(\chi)$  and we define  $\mathcal{B}^*$  expanding  $(\mathcal{H}(\chi), \in)$  by  $P_1 = |M_1|, P_2 = |M_2|, P = |M|, Q_0 = \Phi, <_1 = <^{M_2}, < = <_1 \upharpoonright P, m = a$

constant symbol for a set  $M_1, R^{\mathcal{B}^*} = R^{M_1}$  for  $R \in \tau_T$  (wlog  $\tau(T)$  does not contain any other predicate mentioned here)

$$Q = \{ \langle \langle \varphi_\ell(x, \bar{a}_\ell) : \ell < n \rangle : M_1 \models \exists x [\wedge \varphi_\ell(x, \bar{a}_\ell)] \}.$$

$H$  is a partial unary function with domain  $Q$  and range  $P_1$ ,  $H(\langle \varphi_\ell(x, \bar{a}_\ell) : \ell < n \rangle)$  satisfies  $\{ \varphi_\ell(x, \bar{a}_\ell) : \ell < n \}$ , i.e.  $\mathcal{B}^*$  satisfies the formula “ $m \models (\exists x) \bigwedge_{\ell < n} \varphi_\ell(x, \bar{a}_\ell)$ ”.

Let  $T^* = Th(\mathcal{B}^*)$ , let  $\bar{\varphi}_1$  be the trivial interpretation of  $T$  in  $T^*$  (the restriction + reduct) and  $\bar{\varphi}_2 = \langle P_2(x), x_0 <_1 x_1 \rangle$  is an interpretation of  $T_{tr}^*$ . So  $T^*, \bar{\varphi}_1, \bar{\varphi}_2$  does not depend on  $\lambda$ .

Now we assume  $\mathcal{B}$  is a model of  $T^*$ ,  $N_1 = \mathcal{B}^{[\bar{\varphi}_1]}$ ,  $N_2 = \mathcal{B}^{[\bar{\varphi}_2]}$ ,  $N_3 = (P^{\mathcal{B}}, <^{\mathcal{B}})$  and we aim to show that (i) below implies (iii). We will first show that (i)  $\Rightarrow$  (ii) and use this fact in the proof.

- (i)  $N_2$  is  $\lambda$ -saturated
- (ii) in  $N_3$  every branch has cofinality  $\geq \lambda$ , equivalently: every increasing sequence of length  $< \lambda$  has an upper bound
- (iii)  $N_1$  is  $\lambda$ -saturated.

Why (i)  $\Rightarrow$  (ii)? If  $\langle a_i : i < \delta \rangle$  is  $<^{N_3}$ -increasing,  $\delta < \lambda$  then it is  $<^{N_2}$ -increasing hence has a  $<^{N_2}$ -upper bound  $a$  but  $(\forall x \in P_2)(\exists y)(x <_1 y \wedge P(y))$  belongs to  $T^*$  so there is  $b, a <^{N_2} b \in P^N = N_3$  so  $b$  is as required.

So we can assume clause (i) and we shall prove (iii).

Before we proceed, let us note several trivial but important properties of  $\mathcal{B}$ .

- (a) We can talk inside  $\mathcal{B}$  about a set being a model, (standard coding of) a formula, a proof, etc. In particular, we can speak about  $m$  (as a model) satisfying or not satisfying certain sentences. Also, given a formula with free variables we can speak about substitution of other variables or parameters into the formula. Given  $s \in \mathcal{B}$  which is a formula with free variables  $\bar{x}$ , we will allow ourselves to write  $s = s(\bar{x})$ , and if  $\mathcal{B}$  thinks that substitution of  $\bar{a} \in P_1$  into  $s$  will turn it into a true sentence in  $m$  as a model, we will write  $m \models s(\bar{a})$  or just  $s(\bar{a})$ .
- (b)  $\mathcal{B} \models \forall z Q_0(z) \iff$  “ $z$  is a formula with one free variable with parameters from  $P_1^{\mathcal{B}}$ . Moreover, suppose  $\varphi(x, \bar{a})$  is a formula in  $L(\tau_T)$  s.t.  $\bar{a} \in P_1^{\mathcal{B}}$ .  $\mathcal{B}^*$  and therefore  $\mathcal{B}$  satisfy  $(\forall \bar{y} \in P_1)(\exists! s \in Q_0)$  such that  $(\forall x \in P_1) \varphi(x, \bar{y}) \iff “m \models s(x, \bar{y})”$ . Let us denote by  $\ulcorner \varphi(x, \bar{a}) \urcorner$  this “canonical encoding” of  $\varphi(x, \bar{a})$  in  $Q_0^{\mathcal{B}}$ .
- (c)  $\mathcal{B} \models \forall s P(s) \iff$  “ $s$  is a finite sequence of members of  $Q_0$ , i.e.  $(\exists n \in \omega)(s : n \rightarrow Q_0)”$ .
- (d) For simplicity of notation, given  $s \in P^{\mathcal{B}}$ , we will write “ $z \in s$ ” instead of “ $z \in Im(s^{\mathcal{B}})$ ”.
- (e) For  $z \in P^{\mathcal{B}}$ ,  $c \in P_1^{\mathcal{B}}$ , we write  $z(c)$  meaning  $(\forall s \in z) s(c)$ .
- (f) For every  $\varphi(x, \bar{a}) \in L(\tau_T)$  for  $\bar{a} \in P_1^{\mathcal{B}}$ , there exists an element of  $P^{\mathcal{B}}$  corresponding to the finite sequence  $\langle \varphi(x, \bar{a}) \rangle$ . We denote this element by  $\ulcorner \varphi(x, \bar{a}) \urcorner$ . Moreover,  $\mathcal{B} \models \exists x (P_1(x) \wedge \varphi(x, \bar{a})) \rightarrow Q(\ulcorner \varphi(x, \bar{a}) \urcorner)$ .

*Subclaim 3.7.1.* (1) Suppose  $\mathcal{B} \models Q(z)$ . Then  $\mathcal{B} \models \forall w (Q(w) \wedge z < w) \rightarrow z(H(w))$ .

- (2) Let  $\varphi(x, \bar{a}) \in L(\tau_T)$  and suppose  $\mathcal{B} \models \exists x P_1(x) \wedge \varphi(x, \bar{a})$ . Then  $\mathcal{B} \models \forall z (Q(z) \wedge \ulcorner \varphi(x, \bar{a}) \urcorner < z) \rightarrow \varphi(H(z), \bar{a})$ .

*Proof.* (1) Trivial as  $\mathcal{B}^*$  satisfies it.

- (2) Let  $z^* = \langle \ulcorner \varphi(x, \bar{a}) \urcorner \rangle$ . First,  $Q(z^*)$  holds by  $f$  above. By (1),  $z^*(H(z))$  holds for each  $z \in Q^{\mathcal{B}}$ ,  $z^* < z$ . Now by  $b$  and  $f$  above,  $\mathcal{B} \models \forall x P_1(x) \rightarrow (z^*(x) \iff \varphi(x, \bar{a}))$ . As  $\mathcal{B} \models \text{Range}(H) \subseteq P_1$ , we are done.  $\square$

We now proceed with the proof (i)  $\implies$  (iii). So let  $p$  be a 1-type in  $N_1$  of cardinality  $< \lambda$ , so let  $p = \{\varphi_\beta(x, \bar{a}_\beta) : \beta < \alpha\}$  with  $\alpha < \lambda$ ,  $\bar{a}_\beta \in N_1 \forall \beta$ . Without loss of generality  $p$  is closed under conjunction, i.e. for every  $\varepsilon, \zeta < \alpha$  for some  $\xi < \alpha$  we have  $\varphi_\xi(x, \bar{a}_\xi) = \varphi_\varepsilon(x, \bar{a}_\varepsilon) \wedge \varphi_\zeta(x, \bar{a}_\zeta)$ . We shall now choose by induction on  $\beta \leq \alpha$  an element  $b_\beta$  of  $N$  such that

- (A)  $b_\beta \in P^{\mathcal{B}} = N_3$  moreover  $b_\beta \in Q^{\mathcal{B}}$  and  $\gamma < \beta \Rightarrow b_\gamma <^{N_3} b_\beta$
- (B) if  $\gamma < \beta$  then  $\mathcal{B} \models (\forall z)(Q(z) \wedge (b_\beta \leq z) \rightarrow \varphi_\gamma(H(z), \bar{a}_\gamma))$
- (C) if  $\gamma < \alpha$  (but not necessarily  $\gamma < \beta$ ) then  $\mathcal{B} \models (\exists z)[Q(z) \wedge (b_\beta \leq z) \wedge (\forall y)(Q(y) \wedge z \leq y \rightarrow \varphi_\gamma(H(y), \bar{a}_\gamma))]$ .

If we succeed then  $H^{\mathcal{B}}(b_\alpha)$  is as required.

Case 1:  $\beta = 0$ .

Define  $b_0 = \langle \rangle$  (the element of  $P^{\mathcal{B}}$  corresponding to the empty sequence). Clearly  $\mathcal{B} \models Q(b_0)$ , i.e. the demand (A) holds. (B) holds trivially. Why does (C) hold? Let  $\gamma < \alpha$ .  $\mathcal{B} \models \exists x \varphi_\gamma(x, \bar{a}_\gamma)$  therefore denoting  $z_\gamma^* = \langle \ulcorner \varphi_\gamma(x, \bar{a}_\gamma) \urcorner \rangle$ , we have  $\mathcal{B} \models Q(z_\gamma^*) \wedge b_0 < z_\gamma^*$ . Now we finish by part (2) of the subclaim.

Case 2:  $\beta = v + 1$ .

$\mathcal{B}$  satisfies the sentence saying that for every  $\eta \in Q$  and  $\bar{y} \in P_1$  there exists an element of  $P$  that we denote by  $\text{Conc}_v(\eta, \bar{y})$  corresponding to  $\eta \wedge \langle \ulcorner \varphi_v(x, \bar{y}) \urcorner \rangle$ . We define  $b_\beta = \text{Conc}_v(b_v, \bar{a}_v)$ . Now we have to check (A) - (C).

- (A) By the induction hypothesis, clause (C) holds for  $b_v$  and  $v$  (standing for  $b_\beta$  and  $\gamma$  there). Therefore  $\mathcal{B} \models \exists z \in Q(b_v \leq z) \wedge \varphi_v(H(z), \bar{a}_v)$ . But  $\mathcal{B}^*$  (and so  $\mathcal{B}$ ) satisfies that  $\forall \bar{y} \in P_1$  if there exists  $z \in Q$  s.t.  $\varphi_v(H(z), \bar{y})$  holds, then  $\text{Conc}_v(z, \bar{y})$  is an element of  $Q$  (as in  $\mathcal{B}^*$  the assumption means that there exists an element of  $m$  satisfying all the formulae in  $z$  plus  $\varphi_v(x, \bar{y})$ ). So we get the required.
- (B) is clear as by the induction hypothesis,  $\varphi_\zeta(H(z), \bar{a}_\zeta)$  holds for every  $\zeta < v$ ,  $b_\beta \leq z$  (recall that  $b_v \leq b_\beta$ ). As for  $\varphi_v(x, \bar{a}_v)$ ,  $\mathcal{B}^*$  clearly satisfies that for every  $z \in Q, \bar{y} \in P_1$ , if  $b = \text{Conc}_v(z, \bar{y})$  is in  $Q$  then  $\varphi_v(H(z), \bar{y})$  holds  $\forall z \in Q, b \leq z$ .
- (C) Let  $\zeta < \alpha$ . As  $p$  is closed under conjunctions, for some  $\xi$ ,  $\varphi_\gamma(x, \bar{a}_\gamma) \wedge \varphi_\zeta(x, \bar{a}_\zeta) = \varphi_\xi(x, \bar{a}_\xi)$ . Now we apply clause (C) holding for  $b_v$  to  $\gamma = \xi$  and get  $z \in Q, b_v \leq z$  with  $H(z)$  satisfying both  $\varphi_v(x, \bar{a}_v)$  and  $\varphi_\zeta(x, \bar{a}_\zeta)$ . Once again using the satisfaction by  $\mathcal{B}$  of natural sentences, we show that  $b = \text{Conc}_\zeta(b_\beta, \bar{a}_\zeta)$  is in  $Q$ ,  $b_\beta \leq b$  and  $\forall z \in Q$  which is above  $b$ ,  $\varphi_\zeta(x, \bar{a}_\zeta)$  holds, i.e.  $b$  is as required.

Case 3:  $\beta = \delta$  limit.

By our present assumption, clause (i), and therefore clause (ii), hold. Hence there is  $b \in P^{\mathcal{B}}$  which is an upper bound to  $\{b_\gamma : \gamma < \beta\}$ . Now  $\mathcal{B}$  satisfies “for every element  $z$  of  $P$  there is a  $y \leq z$  which is in  $Q$  and  $x \leq z \wedge Q(x) \rightarrow x \leq y$ ”. Apply this to  $b$  for  $z$  and get  $b'_\delta$  for  $y$ . So  $b'_\delta \in Q$  and  $\gamma < \delta \Rightarrow b_\gamma \leq b'_\delta$ , as required in clauses (A) + (B) but not necessarily (C).

Define for each  $\zeta < \alpha$  a formula  $\psi_\zeta(w, \bar{a}_\zeta) = (\exists z)(w \leq z \wedge Q(z) \wedge (\forall y)(z \leq y \wedge Q(y) \rightarrow \varphi_\zeta(H(y), \bar{a}_\zeta))$  Now we find  $c_\zeta$  (for  $\zeta < \alpha$ ) such that:

- (a)  $c_\zeta \in Q^{\mathcal{B}}, c_\zeta \leq b$
- (b)  $\psi_\zeta(c_\zeta, \bar{a}_\zeta)$  holds.
- (c) under (a) + (b), the element  $c_\zeta$  is maximal.

Why do  $c_\zeta$  exist?  $\mathcal{B}$  satisfies “for every element  $s$  of  $P$  there is a  $w \leq s$  which satisfies  $\psi_\zeta(w, \bar{a}_\zeta)$ , is in  $Q$  and  $(x \leq s \wedge \psi_\zeta(x, \bar{a}_\zeta) \wedge Q(x)) \rightarrow (x \leq w)$ ”.

By the induction hypothesis we have:

$$\gamma < \delta, \zeta < \alpha \Rightarrow b_\gamma <^{N_3} c_\zeta.$$

Clearly it suffices to find  $b_\delta$  satisfying  $Q(b_\delta)$  and  $b_\gamma <^{N_3} b_\delta <^{N_3} c_\zeta$  for  $\gamma < \delta, \zeta < \alpha$ . As  $N_3 \upharpoonright \{c : c \leq b\}$  is linearly ordered, this follows from  $N_2$  being  $\lambda$ -saturated.  $\square$

**Proposition 3.8.** (1) For every  $T^*$ , there is  $T^{**} \supseteq T^*, |T^{**}| = |T^*| + \aleph_0$  such that for every model  $\mathcal{B}$  of  $T^{**}$  we have

- (a) for any  $\lambda$ , the following are equivalent
  - ( $\alpha$ ) if  $\bar{\varphi}_1$  is an interpretation of  $T_{tr}^*$  in  $\mathcal{B}$  (possibly with parameters) then  $\mathcal{B}^{[\bar{\varphi}_1]}$  is  $\lambda_{tr}$ -saturated
  - ( $\beta$ ) if  $\bar{\varphi}_2$  is an interpretation of  $T_{ord}$  in  $\mathcal{B}$  (possibly with parameters) then  $\mathcal{B}^{[\bar{\varphi}_2]}$  is  $\lambda$ -saturated
- (b) for any  $\lambda$ , the following are equivalent
  - ( $\alpha$ ) if  $\bar{\varphi}_1$  is an interpretation of  $T_{tr}$  in  $\mathcal{B}$  (possibly with parameters) then in  $\mathcal{B}^{[\bar{\varphi}_1]}$ , every branch with no last element has cofinality  $\geq \lambda$
  - ( $\beta$ ) if  $\bar{\varphi}_2^*$  is an interpretation of  $T_{ord}$  in  $\mathcal{B}$  (possibly with parameters) then in  $\mathcal{B}^{[\bar{\varphi}_2]}$  there is no Dedekind cut  $(I_1, I_2)$  with both cofinalities  $< \lambda$  and at least one  $\geq \aleph_0$ .

*Proof.* Easy.  $\square$

**Corollary 3.9.** (1)  $T_{ord}^*$  is  $\triangleleft_\lambda^*$ -maximal.  
 (2) If  $|T| < \lambda$  and  $T$  has  $SOP_3$  then  $T$  is  $\triangleleft_\lambda^*$ -maximal.

*Proof.* (1) Follows from 3.8

(2) By (1) and 3.5.  $\square$

*Question 3.10.* Is the other direction of 3.9 (2) true?

*Remark 3.11.* We present later a proof of a weaker version of the other direction: we get  $SOP_2$  instead of  $SOP_3$ .

We would like to prove a result similar to 3.5 for  $SOP_2$  (or to show maximality in some other way), but unfortunately right now we only can present the following local theorem:

**Theorem 3.12.** If  $T$  has  $SOP_2$  as exemplified by  $\vartheta(\bar{x}; \bar{y})$ , then  $(T_{tr}^*, \vartheta_{tr}(x; y)) \triangleleft_\lambda^* (T, \vartheta(\bar{x}; \bar{y}))$  for any  $\lambda \geq |T| + \aleph_0$  regular.

*Proof.* We can find a model  $M_1$  of  $T_{tr}^*$  and model  $M_2$  of  $T$  and  $\bar{a}_b \in {}^{\ell g(\bar{y})} M_2$  for  $b \in M_1$  such that:

- ( $\alpha$ ) if  $M_1 \models b_0 < \dots < b_{n-1}$  then  $\{\vartheta(\bar{x}, \bar{a}_{b_\ell}) : \ell < n\}$  is satisfiable in  $M_2$
- ( $\beta$ ) if  $b_1, b_2$  are incomparable in  $M_1$  then

$$M_2 \models \neg(\exists \bar{x})(\vartheta(\bar{x}, \bar{a}_{b_1}) \& \vartheta(\bar{x}, \bar{a}_{b_2}))$$

- ( $\gamma$ ) for no  $\bar{d} \in {}^{\ell g(\bar{x})}(M_2)$  is  $\{b \in M_1 : M_2 \models \vartheta(\bar{d}, \bar{a}_b)\}$  unbounded in  $M_1$  (note that by ( $\beta$ ) it is always linearly ordered in  $M_1$ , therefore ( $\gamma$ ) means that for each  $\bar{d} \in {}^{\ell g(\bar{x})}(M_2)$ , there exists an element of  $M_1$  which is above every  $b$  satisfying  $\vartheta(\bar{d}, \bar{a}_b)$ ).

[How? Choose by induction on  $n$ ,  $(M_{1,n}, M_{2,n}, \langle \bar{a}_b : b \in M_{1,n} \rangle : n < \omega)$  such that:

- (a)  $M_{1,n}$  is a model of  $T_{tr}^*$
- (b)  $M_{2,n}$  is a model of  $T$
- (c)  $M_{1,n} \prec M_{1,n+1}$  moreover, every branch of  $M_{1,n}$  has an upper bound in  $M_{1,n+1}$
- (d)  $M_{2,n} \prec M_{2,n+1}$
- (e)  $\bar{a}_b \in {}^{\ell g(\bar{y})}(M_{2,n})$  for  $b \in M_{1,n}$
- (f) clauses ( $\alpha$ ), ( $\beta$ ) hold
- (g) if  $b \in M_{1,n+1}$  and  $[b' \in M_{1,n} \Rightarrow M_{1,n+1} \models \neg(b < b')]$  then  $\vartheta(\bar{x}, \bar{a}_b)$  is not satisfied by any sequence from  $M_{1,n}$ .

There is no problem to carry the definition.

Now  $M_1 = \bigcup_n M_{1,n}$ ,  $M_2 = \bigcup_n M_{2,n}$  and  $\langle \bar{a}_b : b \in M_1 \rangle$  are as required above.]

Now let  $\chi$  be such that  $M_1, M_2 \in \mathcal{H}(\chi)$ , wlog  $\tau_T = \tau(M_2)$ ,  $\{<\} = \tau(T_{tr}) = \tau(M_1)$  and  $\{\in\}$  are pairwise disjoint. Now we define a model  $\mathcal{B}_0$ .

Its universe is  $\mathcal{H}(\chi)$  relation  $\in$  (membership)

$$P_1 = |M_1|,$$

$$P_2 = |M_2|$$

$R = R^{M_\ell}$  if  $R \in \tau(M_\ell)$ ,  $\ell \in \{1, 2\}$   $F_\ell$  (for  $\ell < \ell g(\bar{y})$ ) a partial unary function such that:  $b \in M_1 \Rightarrow \langle F_\ell(b) : \ell < \ell g(\bar{y}) \rangle = \bar{a}_b$ .

Let  $T^* = Th(\mathcal{B}_0)$ . For the obvious  $\bar{\varphi}$  and  $\bar{\psi}$ ,  $T^*$  is  $(T, T_{tr})$ -superior and  $|T^*| = |T| + \aleph_0$ . Assume  $\lambda = \text{cf}(\lambda) > |T^*|$ .

So let  $\mathcal{B}$  be a model of  $T^*$  such that  $M'_2 = \mathcal{B}^{[\bar{\varphi}]}$ , the model of  $T$  interpreted in it, is  $\lambda^+$ -saturated. It will be enough to prove that  $M'_1 = \mathcal{B}^{[\bar{\psi}]}$  satisfies: for every branch of cofinality  $\theta \leq \lambda$  there exists an upper bound. So let  $\{b_i : i < \theta\}$  be  $<^{M_1}$ -increasing let  $\bar{c}_i = \langle F_\ell^{\mathcal{B}}(b_i) : \ell < \ell g(\bar{y}) \rangle$ . Hence for any  $n < \omega$ ,  $i_0 < \dots < i_{n-1} < \theta$  we have  $M'_2 \models (\exists \bar{x})[\bigwedge_{m < n} \vartheta(\bar{x}, \bar{c}_{i_m})]$  because  $\mathcal{B}_0 \models (\forall z_0, \dots, z_{n-1})[\bigwedge_{k < n} P_1(z_k) \Rightarrow z_0 < z_1 < \dots < z_{n-1} \rightarrow (\exists \bar{x}) \bigwedge_{m < n} \vartheta(\bar{x}, \langle F_\ell(z_m) : \ell < \ell g(\bar{y}) \rangle)]$ .

So  $\{\vartheta(\bar{x}, \bar{c}_i) : i < \theta\}$  is finitely satisfiable in  $M'_2$  hence some  $\bar{d} \in {}^{\ell g(\bar{x})}(M'_2)$  realizes it. Now we claim that  $\{b \in M'_1 : \mathcal{B} \models \vartheta(\bar{d}, \bar{a}_b)\}$  is bounded in  $M'_1$ . Why? Recall that by clause ( $\gamma$ )  $\mathcal{B}_0$  satisfies: for every  $\bar{x} \in {}^{\ell g(\bar{y})}P_2$  there exists  $z \in P_1$  such that  $z$  is  $<^{\mathcal{B}}$ -above all the elements  $w \in P_1$  satisfying  $\vartheta(\bar{x}, \bar{a}_w)$ . Therefore  $\mathcal{B}$  satisfies this sentence, and applying it to  $\bar{d} \in {}^{\ell g(\bar{x})}(M'_2)$ , we get  $b^* \in M'_1$  - the required bound. As for each  $i < \theta$ ,  $\vartheta(\bar{d}, \bar{a}_{b_i})$  holds, clearly  $\mathcal{B} \models b_i < b^*$  for all  $i$ , and we are done.  $\square$

The next goal is to complete the proof started in [DjSh692] of the fact that  $\triangleleft^*$ -maximality implies  $SOP_2$ . In [DjSh692] a property was defined -  $\triangleleft_\lambda^{**}$ -maximality, which is closely related to  $\triangleleft_\lambda^*$ -maximality and it was shown in theorem (3.4) that every  $T$  which is  $\triangleleft_\lambda^{**}$ -maximal for some (every) big enough regular  $\lambda$ , has an order

property similar to  $SOP_2$ , that we call  $SOP_2''$  (see 1.4). We answer the question (3.8)(3) from [DjSh692] showing that  $SOP_2''$  is equivalent to  $SOP_2$  (for a theory).

So assuming that  $T$  is  $\triangleleft_{\lambda^+}^*$ -maximal for some regular  $\lambda$  satisfying  $2^\lambda = \lambda^+$ , we get by [DjSh692], claim (3.2),  $T$  is  $\triangleleft_{\lambda^*}^{**}$ -maximal, so it has  $SOP_2''$ , and therefore  $SOP_2$ .

**Theorem 3.13.** *Let  $T$  be a theory.*

- (1) *Suppose  $\vartheta(\bar{x}, \bar{y})$  exemplifies  $SOP_2$  in  $T$ . Then  $\vartheta(\bar{x}; \bar{y})$  exemplifies  $SOP_2''$  in  $T$  as well.*
- (2) *Suppose  $\vartheta(\bar{x}, \bar{y})$  exemplifies  $SOP_2''$  in  $T$ . Then for some  $k$ ,  $\vartheta^{<k>}(\bar{x}; \bar{y})$  exemplifies  $SOP_2$  in  $T$  (where  $\vartheta^{<k>}(\bar{x}; \bar{y}) = \bigwedge_{\ell < k} \vartheta(\bar{x}; \bar{y}_\ell)$ ).*

*Proof.* (1) is easy.

- (2) Denote  $\mathcal{J}_\lambda^n = \{\bar{\eta} : \bar{\eta} = \langle \eta_\ell : \ell \leq n \rangle, \eta_\ell \triangleleft \eta_{\ell+1}; \text{ and } \eta_\ell \in {}^{\lambda > 2}\}$ . So assume  $\vartheta(\bar{x}; \bar{y})$  has  $SOP_2''$  as exemplified by  $n, \bar{\mathbf{a}} = \langle a_{\bar{\eta}} : \bar{\eta} \in \mathcal{J}_\omega^n \rangle$ . Without loss of generality  $\langle \bar{a}_{\bar{\eta}} : \bar{\eta} \in \mathcal{J}_\omega^n \rangle$  is tree indiscernible in the relevant sense:  $\eta \smallfrown \langle 0 \rangle$ ,  $\eta \smallfrown \langle 1 \rangle$  look the same over  $\eta$  (2-fbti from 1.7). We can assume this by 1.8 (for more details, see [DjSh692], claim (2.14)).

For  $\nu \in {}^\omega \geq 2$  let  $p_\nu = \{\vartheta(\bar{x}, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta_\ell : \ell < n \rangle, \eta_\ell < \eta_{\ell+1} \leq \nu\}$  so

$\otimes_1$   $p_\eta$  for  $\eta \in {}^\omega 2$  is consistent (in  $\mathfrak{C}_T$ ).

Let

$$\Xi = \{(h, \Upsilon) : h \text{ is a one-to-one mapping from } {}^{n \geq m} \text{ to } {}^{\omega > 2} \text{ preserving } \triangleleft, \perp \text{ and } \Upsilon \subseteq {}^n m \text{ and there is } \langle \nu_\eta^* : \eta \in \Upsilon \rangle, h(\eta) \triangleleft \nu_\eta^* \in {}^\omega 2 \text{ for } \eta \in {}^n m \text{ such that } \cup \{p_{\nu_\eta^*} : \eta \in \Upsilon\} \text{ is inconsistent}\}.$$

Now

$\otimes_2$   $\Xi$  is nonempty

[Why? By the definition of  $SOP_2''$ , clause (b), choose  $\Upsilon = {}^n m$ ]

Choose  $(h^*, \Upsilon^*) \in \Xi$  with  $|\Upsilon^*|$  of minimal cardinality and  $\langle \nu_\eta^* : \eta \in \Upsilon^* \rangle$  as there. By  $\otimes_1$  clearly  $|\Upsilon^*| \geq 2$ . So choose  $\eta_0 \neq \eta_1$  from  $\Upsilon^*$  with  $\nu^* = \nu_{\eta_0}^* \cap \nu_{\eta_1}^*$  ( $= h(\eta_0) \cap h(\eta_1)$ ) being of maximal ???? length and let  $k^* = \ell g(\nu^*)$ . We can find  $\ell^* < \omega$  sufficiently large such that  $\cup \{p_{\nu_\eta^* \upharpoonright \ell^*} : \eta \in \Upsilon^*\}$  is inconsistent. We choose by induction on  $i < \omega$  for every  $\rho \in {}^\ell 2$ , a sequence  $\nu_\rho \in {}^{\omega > 2}$  by  $\nu_{<>} = \nu^*, \nu_{\rho \smallfrown \langle j \rangle} = \nu_\rho \hat{\smallfrown} h(\eta_j)$ .

Lastly for  $\rho \in {}^{\omega > 2} \in \{<>\}$  let  $\vartheta^*(\bar{x}, \bar{b}_\rho)$  be the conjunction of

$$\begin{aligned} & \cup \{p_{\nu_\eta^* \upharpoonright \ell^*} : \eta \in \Upsilon^* \setminus \{\eta_0, \eta_1\}\} \cup \{\vartheta(\bar{x}, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta_\ell : \ell \leq n \rangle, \\ & \eta_\ell \triangleleft \eta_{\ell+1} \leq \nu_\rho \text{ and } (\forall \ell \leq n) [\ell g(\eta_\ell) \notin [k, \ell g(\nu_\rho) - \ell^*]] \\ & \text{(this condition is empty if } \ell g(\rho) = 1)\}. \end{aligned}$$

Now if  $\rho^* \in {}^\omega 2$  then  $\{\vartheta^*(\bar{x}, \bar{b}_\rho) : \rho \triangleleft \rho^*\}$  is consistent as all its members are conjunctions of formulas from

$$\cup \{p_{\nu_\eta^*} : \eta \in \Upsilon^* \setminus \{\eta_0^*, \eta_1^*\}\} \cup p_{\rho^*}$$

and this is consistent as otherwise  $(h^* \upharpoonright (\Upsilon^* \setminus \{\eta_0^*, \eta_1^*\})) \cup \{\langle \eta_0^*, \rho^* \upharpoonright \ell^{**} \rangle\}$ ,  $\Upsilon^* \setminus \{\eta_1^*\}$  belongs to  $\Xi$  for some  $\ell^{**}$ , thus contradicting the choice of  $(h^*, \Upsilon^*)$ , i.e. with minimal  $|\Upsilon^*|$ .

Lastly if  $\rho_0, \rho_1 \in {}^{\omega > 2}$  are  $\triangleleft$ -incomparable then  $\{\vartheta^*(\bar{x}; \bar{b}_{\rho_0}), \vartheta^*(\bar{x}; \bar{b}_{\rho_1})\}$  is inconsistent: we know that

⊗ 3.13.1.

$$\bigcup \{p_{\nu_{\eta}^* \upharpoonright \ell^*} : \eta \in \Upsilon^* \setminus \{\eta_0, \eta_1\}\} \cup \{\vartheta(\bar{x}, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta_\ell : \ell \leq n \rangle, \\ \eta_\ell \triangleleft \eta_{\ell+1} \trianglelefteq \nu_{\eta_0}^* \upharpoonright \ell^*\} \cup \{\vartheta(\bar{x}, \bar{a}_{\bar{\eta}}) : \bar{\eta} = \langle \eta_\ell : \ell \leq n \rangle, \eta_\ell \triangleleft \eta_{\ell+1} \trianglelefteq \nu_{\eta_1}^* \upharpoonright \ell^*\}.$$

is inconsistent (by the choice of  $(h^*, \Upsilon^*) \in \Xi$  and the choice of  $\ell^*$ ). Now, by the fact that  $\nu^* = \nu_{\eta_0}^* \cap \nu_{\eta_1}^*$  was chosen to be maximal among other pairs in  $\Upsilon^*$ , we see that if

$$\bar{\eta}_0 = \langle \eta_\ell^0 : \ell \leq n \rangle, \text{ where for each } \ell, \eta_\ell^0 \triangleleft \eta_{\ell+1}^0 \trianglelefteq \nu_{\eta_0}^* \upharpoonright \ell^*$$

and

$$\bar{\eta}_1 = \langle \eta_\ell^1 : \ell \leq n \rangle, \text{ where for each } \ell, \eta_\ell^1 \triangleleft \eta_{\ell+1}^1 \trianglelefteq \nu_{\eta_1}^* \upharpoonright \ell^*$$

while

$$\bar{\eta}_3 = \langle \eta_\ell^3 : \ell \leq n \rangle, \text{ where for each } \ell, \eta_\ell^2 \triangleleft \nu_{\eta^*}^* \text{ for some } \eta^* \in \Upsilon^* \setminus \{\eta_0^*, \eta_1^*\}$$

then

⊗ 3.13.2.

$$\bar{\eta}_1 \frown \bar{\eta}_2 \frown \bar{\eta}_3 \equiv \bar{\varsigma}_1 \frown \bar{\varsigma}_2 \frown \bar{\eta}_3$$

where  $\bar{\varsigma}_j = \langle \varsigma_\ell^j : \ell \leq n \rangle$  and

$$\varsigma_\ell^j = \eta_\ell^j, \text{ if } \lg(\eta_\ell^j) \leq k^* \\ \varsigma_\ell^j = \nu_{\rho_j} \upharpoonright [\lg(\nu_{\rho_j}) - (\ell^* - \lg(\eta_\ell^j))], \text{ otherwise}$$

In simpler words: we replace every  $\eta_\ell^j$  (an initial segment of  $\nu_{\eta_j} \upharpoonright \ell^*$ ) whose length is bigger than  $k^*$  (in particular, it is not below any element in the image of  $\Upsilon^*$  other than  $\nu_{\eta_j}$  itself) by an appropriate initial segment of  $\nu_{\rho_j}$ , and get a similar sequence over the image of  $\Upsilon^* \setminus \{\eta_0^*, \eta_1^*\}$ .

Now, by indiscernibility of  $\langle \bar{a}_{\bar{\eta}_\alpha} \rangle$ , the definition of  $\vartheta^*(\bar{x}, \bar{b}_\rho^*)$ , 3.13.1 and 3.13.2, we conclude  $\{\vartheta^*(\bar{x}; \bar{b}_{\rho_0}), \vartheta^*(\bar{x}; \bar{b}_{\rho_1})\}$  is also inconsistent.  $\square$

## REFERENCES

- [DjSh614] M.Džamonja and S.Shelah, *On the existence of universals and an application to Banach spaces*, to appear in the Israel Journal of Mathematics.
- [DjSh692] M.Džamonja and S.Shelah, *On  $\triangleleft^*$ -maximality*, submitted
- [Sh88] S.Shelah, *Classification of nonelementary classes. II. Abstract elementary classes*, in Classification Theory (Chicago, IL 1985), 419-497, Lecture Notes in Mathematics, Springer, 1987.
- [Sh457] S.Shelah, *The Universality Spectrum: Consistency for more classes in Combinatorics, Paul Erdős is Eighty*, Vol. 1, 403-420, Bolyai Society Mathematical Studies, 1993, Proceedings of the Meeting in honor of P. Erdős, Keszthely, Hungary 7. 1993, an improved version available at <http://www.math.rutgers.edu/~shelaharch>
- [Sh500] S.Shelah, *Towards classifying unstable theories*, Annals of Pure and Applied Logic 80 (1996) 229-255.
- [Sh:c] S.Shelah, *Classification theory and the number of nonisomorphic models*. Volume 92 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1990.

MATHEMATICS DEPARTMENT, HEBREW UNIVERSITY OF JERUSALEM, 91904 GIVAT RAM, ISRAEL,